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BIOGRAPHY.

DR. GEORGE BRUCE HALSTED.

BY LEONARD E. DICKSON, M. A.

DR. HALSTED is of a historic family, a descendant of the second son of Sir Lawrance Halsted, Admiral of the British Navy, to whom the Halsted coat-of-arms was granted by Royal Letters-Patent. Dr. Halsted and his brother are Princeton graduates, as were also his father and his father's brother, his grandfather and his grandfather's brother, and his great-grandfather. General Halsted's munificent gift, the Halsted Observatory at Princeton, costing over one hundred thousand dollars, is a fitting memorial of this connection through generations.

The family bore an active part in the American Revolution. Dr. Robert Halsted was betrayed to the English and imprisoned in the Old-Sugar-House prison in New York City. While the Halsted Manor was untenanted save by Miss Nancy Halsted and her mother, this young girl discovered a party of British soldiers in an open boat approaching the landing. Giving the alarm to her mother, the young lady seized a heavy musket, and, resting it on the corner of a stone wall, fired straight into the loaded boat just as it was touching shore. Panic-stricken at the resounding report of the heavy musket so utterly unexpected, the soldiers rowed away in haste. The heroic act of this girl had saved many homes from pillage. At a dinner party soon after, General Washington proposed a particular toast to Miss Nancy Halsted for her brave act. Other instances in the family history of such deeds of daring might be mentioned.

Dr. Halsted's mother was the only daughter of a very wealthy family resident in Charleston, South Carolina; but from the first war-trump of 1860 this fabric of wealth, slaves and personal property, began to dissolve, and even


George Bruce Halsted

the family real-estate in Charleston was so depreciated in value that it could scarcely be sold for more than the accumulated taxes.

Thus the subject of our sketch, instead of being a wealthy slave-holder, found himself a poor student at the ancestral college, but unlike his predecessors, dependent solely upon his wits for maintenance. While yet a Freshman, two of his own classmates made the discovery that he possessed an occult power of imparting mathematics to the most obtuse. They found that, by some mysterious and wholly unaccountable gift, he makes mathematical acquirement not only easy, but a delightful recreation, even to those who before had maintained that they never could possibly learn any mathematics. These two classmates, out of friendship for whom young Halsted had voluntarily exerted his gift, now sounded his praises abroad, and he was besought to give lessons for pay to sons of rich men, athletes, rowers, base-ball and foot-ball players in danger of being dropped for failure in mathematics. Not even charging the highest prices could shake off these suppliants for what they considered a sort of mesmeric cure, which received them as acknowledged dunces and graduated them respectable mathematicians.

This enviable reputation, so early attained, has followed Professor Halsted throughout his whole career. Young ladies continually bring their friends into his larger classes, knowing that they will be fascinated by his brilliant lectures, remarkable alike for their clear, lucid, rigorous style of expounding the intricacies of mathematics, and for their wealth of highly interesting and instructive forays into nearly every field of knowledge.

Young Halsted took first place in his class in mathematics every term of every year of the entire Princeton course, and on graduating, won the mathematical fellowship of six hundred dollars. To perfect himself in applied mathematics, he went from Princeton to Columbia College School of Mines. Here on entering he passed the entire course of pure mathematics, much to the astonishment of the amiable professors, and his tuition fee of two hundred dollars was remitted to him.

While at the school of Mines, in an Inter-collegiate contest open to all American Colleges, Halsted won a prize of two hundred dollars. Among the contestants was Professor Thomas Craig, now head professor of mathematics at Johns Hopkins University. The examiners in this contest were Professors Michie of West Point, and Professor Simon Newcomb of Washington.

Made one of the first twenty Fellows in the new Johns Hopkins University, Halsted constituted alone the first class there of the renowned Sylvester. From their very first meeting the famous Englishman showed a marked friendship for the young American, and his historic address before the University contains a generous tribute to his enthusiastic pupil. When later Dr. Halsted went to Berlin to pursue his studies, Sylvester gave him a flattering letter to Borchardt, then not only one of the four great professors at that greatest of German Universities, but also editing Crelle's Journal and of great private wealth, living in a veritable palace. Since his return to England as Savilian Professor of Geometry in the University of Oxford, Sylvester has

shown that he still cherishes this particular regard for Professor Halsted by proposing his name for membership in the London Mathematical Society.

After two years at Johns Hopkins as Fellow, Halsted received in 1879 the degree Doctor of Philosophy. About this time appeared in three papers in the American Journal of Mathematics his extraordinarily influential Bibliography of Hyper-Space and Non-Euclidean Geometry, since continually referred to in every learned country in the world. This has been in great part reprinted twice—once at Kiev and once in the collected works of Lobachevsky, and a new edition has just been prepared by the eminent Russian geometer, Professor A. Vasiliev, of the University of Kasan.

At this time Dr. Halsted was called to Princeton to plan and inaugurate a system of Post-Graduate Instruction in mathematics. His phenominal success is shown by the fact that his pupils in Graduate Mathematics have become Professor Fine and Professor Magie of Princeton, Professor Kimball of Johns Hopkins, Professor Durell of Dickenson, Professor McNiel of Lake Forest, Professor Perrine and Professor Carman of Leland Stanford, Professor Crew of Northwestern, Professor Riggs and Professor West of Syria, not to enumerate others.

From this brilliant field of work Dr. Halsted was called by a telegram from the University of Texas, announcing his election to the professorship of mathematics, and asking if he would accept the urgent call.

During the period of eight years from his entrance to Johns Hopkins to his acceptance in 1884 of the chair of Pure and Applied Mathematics in the University of Texas, Dr. Halsted produced as many as 17 scientific papers on Mathematics and Logic for leading journals of this country and Eng. as well as his *Metrical Geometry*. This first book from the pen of Dr. Halsted was most favorably received here and in England, and had the honor of being freely drawn upon for the article on "Mensuration" in the last edition of the Encyclopaedia Britannica. In this book appeared Dr. Halsted's remarkable two-term prismoidal formulæ.

The past decade, spent in Texas, has abundantly proven the inexhaustible fruitfulness of Dr. Halsted's genius. Not less than 25 valuable productions, including his books on Geometry, belong to this period. His *Elements of Geometry* which appeared in 1885, has passed through six editions. It is here that Dr. Halsted shows himself the profound geometer that he is. With his usual originality and intuitive genius, he has left the traditional ruts of American writers on Geometry, and built up a delightful but rigorus structure, which has already had no small influence upon sound geometrical teaching in this country.

Perhaps there is no production of Professor Halsted which so well reveals the writer as his *Elementary Synthetic Geometry*, published two years ago. Rejecting utterly the only method the world knew for twenty centuries, he makes no use of congruent triangles, using instead the circle from the very beginning. Two-dimensional spherics follows as the second book, an epoch making innovation. Ratio and proportion are treated with all the rigor of Euclid, in noteworthy contrast to the usual American text-book.

Dr. Halsted's translation of the works of the Russian Lobachevsky and the Hungarian Bolyai on Non-Euclidean Geometry have received the thanks and praise of all. A reprint of these has been made by the Imperial University of Japan.

Many of his scientific articles are of a popular nature, appearing in the Monist, Educational Review, Popular Science Monthly, etc. For a bibliography of Prof. Halsted's works the reader is referred to the *Johns Hopkins Bibliographia Mathematica*, pages 40 and 41, where a list of over forty scientific productions of his is given.

The scientific world has not been slow to recognize in Professor Halsted a great mathematician; for among many honors, he has been made a member of the "Circolo Matematico di Palermo," of the London Mathematical Society, of the Mathematical Society of France, of the two best known Scientific Societies of Mexico, of one in Russia, as well as of several in England and America.

Professor Halsted has been eminently successful in his work in the University of Texas. A natural born teacher, he inspires his pupils with a lasting enthusiasm for science, a love for mathematics. So highly interesting are his discourses upon mathematics, interfused with new and attractive ideas from other fields of knowledge, that it is wholly unnecessary to call a roll to assure attendance at class. I am convinced that his pupils, without exception, thoroughly enjoy the course in mathematics under him, although the majority of them enter with a pronounced dislike for it. No wonder they leave fully convinced of the wonderful genius of this man, the central figure of the Texas University.

As a public lecturer, Dr. Halsted has few equals. By his happy command of speech, his sparkling wit and cutting sarcasm, his wealth of illustration, his breadth of thought, and above all the freshness and newness of his ideas—startling perhaps, but pleasing—he never fails to impress and please his audience. He lectures on such subjects as "Dreams," "Suicide," "The Elixir of Life," "Mexico," as well as strictly scientific topics.

Dr. Halsted married a very refined lady, of a highly connected Southern family, the Swearingens. To their three bright children Dr. Halsted is completely devoted.

The alumni and patrons of the young University of Texas greatly appreciate the honor which Dr. Halsted's high achievments have conferred upon the University and especially its school of Mathematics; and join heartily in wishing that another decade may bring equally great returns to the University from its most distinguished professor.

THE SECOND HYPERBOLIC INTEGRAL.

By F. P. MATZ, M. Sc., Ph. D., New Windsor, Maryland.

From $y^2=(e^2-1)(x^2-a^2)$, $dy/dx=x\sqrt{[(e^2-1)/(x^2-a^2)]}$. Hence the hyperbolic arc bounding a segment symmetrical with respect to the major axis of the hyperbola, becomes

$$s=2\int_o^{x'}\sqrt{\left(1+\frac{(e^2-1)x^2}{x^2-a^2}\right)}dx=2\int_o^{x'}\sqrt{\left(\frac{e^2x^2-a^2}{x^2-a^2}\right)}dx\ldots.(1).$$

Transforming (1) by means of the *circular* notation, $x=a\sec\theta$ and $y=b\tan\theta$, we obtain

$$s=2a\int_o^{\theta'}\sec\theta\sqrt{(e^2\sec^2\theta-1)}d\theta,\ =2ae\int_o^{\theta'}\sec^2\theta\sqrt{\left[1-\left(\frac{1}{e^2}\right)\cos^2\theta\right]}d\theta\ldots.(\alpha).$$

Let $c^2=1/e^2$; that is, $c<1$ and $e>1$. The amplitude of an hyperbolic integral may be $\frac{1}{2}\pi$; but the instant the amplitude exceeds $\sec^{-1}(e)$, the integral transcends the confines of the real. From (α), therefore, we have

$$s=2ae\int_o^{\theta'}\left[\sec^2\theta-\frac{c^2}{2}\left(1+\frac{c^2}{4}\cos^2\theta+\frac{c^4}{8}\cos^4\theta+\text{etc.}\right)\right]d\theta$$

$$=2ae\left[\tan\theta-\frac{c^2}{2}\int\left(1+\frac{c^2}{4}\cos^2\theta+\frac{c^4}{8}\cos^4\theta+\text{etc.}\right)d\theta\right]_o^{\theta'},=\mathbf{H}''(c,\theta')\ldots.(2),$$

which is an hyperbolic integral of the second order and expressed by the circular notation.

[Since $dx=a\sec\theta\tan\theta d\theta$ and $dy=b\sec^2\theta d\theta$, the expressions for s in (α) are easily deduced by a *second* method.]

Integrating (2) between the limits of $\theta=0$ and $\theta=\sec^{-1}(e)$,

$$s=2ae\left\{e\sqrt{\left(1-\frac{1}{e^2}\right)}-\frac{1}{2e^2}\left[1+\frac{1}{8e}\left(1+\frac{3}{8e^4}\right)\sqrt{\left(1-\frac{1}{e^2}\right)}\right.\right.$$

$$\left.\left.+\frac{1}{8e^2}\left(1+\frac{3}{8e^2}\right)\sin^{-1}\left(\sqrt{1-\frac{1}{e^2}}\right)\right]\right\},=\mathbf{H}''[c,\sec^{-1}(e)]\ldots.(3),$$

a symmetrical formula giving a very approximate numerical result. Put $a=1$, and $e=2$; then $s=6.25648+$. According to the Lambertian system of notation for *Hyperbolic Functions*, since $x=a\sec\theta=a\cos U$ and $y=a\tan\theta=a\sin U$, we have from (α) the hyperbolic integral,

$$s=2a\int_o^{U'}\cos U\sqrt{(e^2\cos^2U-1)}\times\frac{\sin U dU}{\cos U\sqrt{(\cos^2U-1)}}$$

$$=2a\int_o^{U'}\surd(e^2\cos^2 U-1)dU=2a\surd(e^2-1)\int_o^{U'}\sqrt{\left[1+\left(\frac{e^2}{e^2-1}\right)\sin^2 U\right]}dU. \; ..(\beta).$$

Make $C^2=e^2 / (e^2-1)$; then, after obvious reductions,

$$s=2a\surd(e^2-1)\int_o^{U'}\surd(1+C^2\sin^2 U)dU=2a\surd(e^2-1)\int_o^{U'}\left[1+\frac{C^2}{2}\sin^2 U\right.$$

$$\left.-\frac{C^4}{8}\sin^4 U+\frac{C^6}{16}\sin^6 U-\text{etc.}\right]dU, = \mathrm{H}''(C,\ U');\text{or}\mathrm{H}''[C, \cosh^{-1}(e)]\ldots.(4),$$

which are forms (1) for the *incomplete* and (2) for the *complete-real* hyperbolic-integral of the second order expressed by the hyperbolic notation of Lambert.

[Since $dx=a\sin U dU$ and $dy=b\cos U dU$, the expressions for s in (β) are easily deduced by a *second* method.]

The analogy between θ and U is expressed through the areas of the circular and hyperbolic sectors. This makes θ the hyperbolic amplitude of U; and this amplitude our former teacher, Professor Cayley, tersely denominated the *Gudermannian of U.*

Remembering that Exp. $U=\cos U+\sin U=\sec\theta+\tan\theta$, we have $U=Gd^{-1}(\theta)=\log(\sec\theta+\tan\theta)=\log\tan(\frac{1}{4}\pi+\frac{1}{2}\theta)\ldots.(\nu)$.

From trigonometrical tables, by means of (ν), the value of U as a function of θ can easily be calculated. Obvious operations, also, give

$$s=2\int_o^{y'}\sqrt{\left(1+\frac{y^2}{(e^2-1)(b^2+y^2)}\right)}dy=\frac{2}{\surd(e^2-1)}\int_o^{y'}\sqrt{\left(\frac{e^2y^2+a^2(e^2-1)^2}{y^2+a^2(e^2-1)}\right)}dy$$

$$=2a\int_o^{\theta'}\sec\theta\surd(e^2\sec^2\theta-1)d\theta, =2a\surd(e^2-1)\int_o^{U'}\sqrt{\left[1+\left(\frac{e^2}{e^2-1}\right)\sin^2 U\right]}dU,$$

which, of course, is just as it should be.

THE INSCRIPTION OF REGULAR POLYGONS.

By LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.

CHAPTER II.

To determine the equation upon the solution of which depends the inscription of the regular polygon of n sides.

Let $na=\pi$. Here n is supposed to be odd. Write p for $\frac{n-1}{2}$. Then $\sin pa=\sin(p+1)a\ldots.(3)$.

Now $\frac{\sin pa}{\sin a}=2^{p-1}\cos^{p-1}a-2^{p-3}(p-2)\cos^{p-3}a+2^{p-5}.\frac{(p-3)(p-4)}{1.2}\cos^{p-5}a$
$+\dots+(-1)^m 2^{p-2m-1}.\frac{(p-m-1)(p-m-2)\dots(p-2m)}{1.2.3\dots m}\cos^{p-2m-1}a\dots$

Hence, also, $\frac{\sin(p+1)a}{\sin a}=2^p\cos^p a-2^{p-2}(p-1)\cos^{p-2}a$
$+2^{p-4}.\frac{(p-2)(p-3)}{1.2}\cos^{p-4}a-\dots$

Substituting in equation (3) and reducing,

$$2^p\cos^p a-2^{p-1}\cos^{p-1}a-2^{p-2}.(p-1)\cos^{p-2}a+2^{p-3}.(p-2)\cos^{p-3}a$$
$$+2^{p-4}.\frac{(p-2)(p-3)}{1.2}\cos^{p-4}a-2^{p-5}.\frac{(p-3)(p-4)}{1.2}\cos^{p-5}a-\dots=0.$$

Writing x for $2\cos a$, $x^p-x^{p-1}-(p-1)x^{p-2}+(p-2)x^{p-3}+\frac{(p-2)(p-3)}{1.2}x^{p-4}$
$-\frac{(p-3)(p-4)}{1.2}x^{p-5}-\dots+(-1)^m\frac{(p-m)(p-m-1)\dots(p-2m+1)}{1.2.3\dots m}x^{p-2m}$
$-(-1)^m\frac{(p-m-1)(p-m-2)\dots(p-2m)}{1.2.3\dots m}x^{p-2m-1}\pm\dots=0\quad\dots(4).$

Beginning with $na=2\pi$, 3π, 4π, $\dots p\pi$, in succession, we find that the p roots of (4) are $2\cos\frac{\pi}{n}$, $-2\cos\frac{2\pi}{n}$, $2\cos\frac{3\pi}{n}$, $-2\cos\frac{4\pi}{n}$, $\dots(-1)^{p+1}.2\cos\frac{p\pi}{n}$

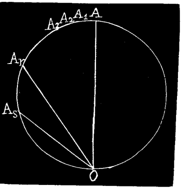
These roots are evidently the chords of OA_1, $-OA_2$, OA_3, $-OA_4$, $\dots(-1)^{p+1}.OA_p$ in a circle of unit radius.

Corollary. Since the sum of the roots of an equation is equal to the negative of the coefficient of the next to the highest power of x, we have the elegant theorem: $OA_1-OA_2+OA_3-OA_4+OA_5-\dots+(-1)^{p+1}.OA_p=1\dots(5)$.

For brevity I will omit the proof* that the general equation (4) is *irreducible*, *i.e.* can not be broken up into equations of lower degree having *rational* coefficients.

The following two theorems are fundamental:

Let the circle of unit radius be supposed divided at A, A_1, A_2, A_3, $\dots A_r$, $\dots A_s$, $\dots$ into n equal parts. Let OA_r and OA_s (henceforth to be written A_r and A_s) denote any two chords. Write $p=\frac{n-1}{2}$.



Theorem I. If the sum of the arcs OA_r and OA_s be *less* than π, $A_r.A_s=A_{s-r}-A_{n-(s+r)}\dots(6)$.

Writing this in its trigonometric form:

*A paper giving this proof was read by the writer before The Texas Academy of Science April 7th, 1894

$2\cos\frac{r\pi}{n}.\cos\frac{s\pi}{n}=\cos(s-r)\frac{\pi}{n}-\cos\frac{(n-s-r)\pi}{n}$, we see that it follows at once from the familiar formula $\cos x-\cos y=2\cos\frac{x+y}{2}\cdot\sin\frac{y-x}{2}$.

Theorem II. If the sum of the arcs OA_r and OA_s be *greater* than π, $A_r.A_s=A_{s-r}+A_{s+r}$ .... (7). Or, otherwise, $2\cos\frac{r\pi}{n}\cdot\cos\frac{s\pi}{n}=\cos\frac{(s-r)\pi}{n}+\cos\frac{(s+r)\pi}{n}$.

We employ Theorem I. when $r+s>p$ and Theorem II. when $not>p$.

When $r=s$, $A_s^2=2+A_{2s}$, when $2s$ is not greater than p; and $A_s^2=2-A_{2s}$, when $2s$ is greater than p.

A quite different (*geometrical*) statement and proof of these theorems is given in Catalan's Geometrie, last edition.

The breaking up into equations of lower degree of the equations upon whose solution we have made the inscription of the regular polygons depend.

It can be proved that the cubics obtained above for the inscription of the regular 7-gon and 9-gon (see chapter I.) can not in any way be avoided or broken up into simpler equations, unless by actually solving them. Likewise the above quintic for the inscription of the regular 11-gon can not be avoided or broken up by our method.

To break up into two cubics the equation $x^6-x^5-5x^4+4x^3+6x^2-3x-1=0$ upon which depends the inscription of the regular polygon of 13 sides.

By theorem (5), $A_1-A_2+A_3-A_4+A_5-A_6=1$. $(A_1+A_3-A_4)(A_5-A_2-A_6)=-3(A_1-A_2+A_3-A_4+A_5-A_6)=-3$, by expanding by equations (6) and (7). Hence, $(A_1+A_3-A_4)$ and $(A_5-A_2-A_6)$ are the two roots of $x^2-x-3=0$. $\therefore A_1+A_3-A_4=\frac{1}{2}(1+\sqrt{13})$; $A_5-A_2-A_6=\frac{1}{2}(1-\sqrt{13})$.

Now $(A_1.A_3-A_1.A_4-A_3.A_4)=-(A_1-A_2+A_3-A_4+A_5-A_6)=-1$.

Also $A_1.A_3.A_4=A_1(A_1-A_6)=2+A_2-A_5+A_6=\frac{1}{2}(3+\sqrt{13})$.

Hence, A_1, A_3, and $-A_4$ are the three roots of the cubic

$$x^3-\tfrac{1}{2}(1+\sqrt{13})x^2-x+\tfrac{1}{2}(3+\sqrt{13})=0.$$

Similarly, $(-A_2.A_5+A_2.A_6-A_5.A_6)=-1$; $A_2.A_5.A_6=\frac{1}{2}(-3+\sqrt{13})$.

Hence, $A_5, -A_2, -A_6$ are the roots of the cubic

$$x^3-\tfrac{1}{2}(1-\sqrt{13})x^2-x+\tfrac{1}{2}(3-\sqrt{13})=0.$$

The product of these two cubics gives the above equation of the sixth degree.

We can make the determination of these six chords depend upon the solution of a single cubic and three quadratics as follows:

As before, $(A_1+A_5)+(A_3-A_2)+(-A_4-A_6)=1$.

$(A_1+A_5)(A_3-A_2)=(-A_1-A_3+A_2+A_4-A_3+A_6+A_2-A_5)$
$=-\{1+(A_3-A_2)\}$.

Similarly, $(A_1+A_5)(-A_4-A_6)=-\{1+(A_1+A_5)\}$; (A_3-A_2) $(-A_4-A_6)=-\{1+(-A_4-A_6)\}$.

$\therefore\ \{(A_1+A_5)(A_3-A_2)+(A_1+A_5)(-A_4-A_6)+(A_3-A_2)$ $(-A_4-A_6)\}=-4$.

Now $(A_1+A_5)(A_3-A_2)(-A_4-A_6)=-(-A_4-A_6)-(A_3-A_2)$ $(-A_4-A_6=A_4+A_6+1+(-A_4-A_6)=1$.

Hence, $(A_1+A_5),(A_3-A_2),(-A_4-A_6)$ are the three roots of the cubic $x^3-x^2-4x-1=0$. Call them A, B, C, respectively.

$$\therefore\ A_1+A_5=A;\ A_1.A_5=(A_4+A_6)=-C.$$
$$A_3-A_2=B;\ -A_3.A_2=-(A_1+A_5)=-A.$$
$$-A_4-A_6=C;\ A_4.A_6=(A_2-A_3)=-B.$$

Hence, A_1 and A_5 are the roots of $x^2-Ax-C=0$.

A_3 and $-A_2$ are the roots of $x^2-Bx-A=0$.

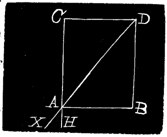
$-A_4$ and $-A_6$ are the roots of $x^2-Cx-B=0$.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton), Ph.D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the September Number.]

Given any triangle (fig.7) ABD right angled at B; prolong DA at any point X, and through A erect HAC perpendicular to AB.



I say the external angle XAH will be equal, or less, or greater than the internal and opposite ADB, according as is true the hypothesis of right angle, or obtuse angle, or acute angle: and inversely.

Proof. Assume in HC the portion AC equal to BD, and join CD. CD will be, in the hypothesis of right angle (P. III.) equal to AB. Wherefore the angle ADB will be equal (Eu. I. 8.) to the angle DAC, or to its equal (Eu. I. 15.) to the angle XAH. Quod erat primo loco demonstrandum.

Then, in the hypothesis of obtuse angle, CD will be (P. III.) less than AB.

Wherefore in the triangle ADB, the angle DAC, or its vertical XAH, will be (Eu. I. 25.) less than the angle ADB. Quod erat secundo loco demonstrandum.

While, in the hypothesis of acute angle, CD will be (P. III.) greater than the opposite AB. Wherefore in the said triangle the angle DAC, or its verticle XAH, will be (Eu. I. 25.) greater than the angle ADA. Quod erat tertio loco demonstrandum.

But inversely: if the angle CAD, or its vertical XAH, be equal to the internal and opposite ADB; the join CD will be (Eu. I. 4.) equal to AB, and therefore the hypothesis of right angle will be (P. IV.) true.

But if however the angle CAD,or its vertical XAH, be less, or greater than the internal or opposite ADB; also the join CD will be (Eu. I. 24.) less or greater than AB; and therefore (P. IV.) will be true respectively the hypothesis of obtuse angle, or acute angle Quod omnia erant demonstranda.

Proposition IX. In any right-angled triangle the two acute angles remaining are taken together equal to one right angle, in the hypothesis of right angle; greater than one right angle, in the hypothesis of obtuse angle; but less in the hypothesis of acute angle.

Proof. For if the angle XAH (fig. 7.) is equal to the angle ADB, which is certain, from the preceding proposition, in the hypothesis of right angle, then the angle ADB makes up with the angle HAD two right angles, as (Eu. I. 13.) the aforesaid angle XAH makes them up with this angle HAB being subtracted, the two angles ADB and BAD remain together equal to one right angle. Quod erat primum.

However, if the angle XAH is less than the angle ADB, which is certain from the preceding proposition, in the hypothesis of obtuse angle, then the angle ADB makes up with the angle HAD more than two right angles, since with it (Eu. I. 13.) the angle XAH makes up two. Wherefore, the angle HAB being subtracted, the two angles ADB and BAD will be together greater than one right angle. Quod erat secundum.

Finally, if the angle XAH be greater than the angle ADB, which is certain from the preceding proposition in the hypothesis of acute angle, then the angle ADB will make up less than two right angles with the angle HAD, since with this (Eu. I. 13.) the angle XAH makes up two. Wherefore, subtracting the right angle HAB, the angles ADB and BAD will be together less than one right angle. Quod erat tertium.

ISOPERIMETRY WITHOUT CURVES OR CALCULUS.

By P H PHILBRICK, M. S., C. E., Lake Charles, Louisiana.

The object of what follows, is to demonstrate by elementary plane geometry, the main propositions of isoperimetry, not using either Steiner's methods, by means of curved figures, or the higher mathematics. For the sake of measurable completeness the usual elementary propositions are included.

PROPOSITION I. *Of all triangles having the same base and equal areas that which is isoceles has the least perimeter.*

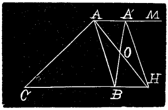


FIG. 1.

Let ABC be an isosceles triangle, and $A'BC$ any other triangle having the same base BC and an equal area.

Since the triangles have the same base and equal areas, their altitudes are equal, and therefore A' is on a line AM, parallel to BC.

Prolong CB to H, making $BH=AA'$ and draw $A'H$ and AH. O is the intersection of the diagonals $A'B$ and AH of the parallelogram $AA'BH$.

Now $AO+OB>AB$, and doubling gives, $AH+A'B>A'H+AB$.

Evidently, $AH=A'C$. $\therefore$ $AH+AB=A'B+A'C$;

Also, $A'H=AC$. $\therefore$ $A'H+AB=AB+AC$.

Hence, $AB+AC<A'B+A'C$;

Also, $AB-A'B<A'C-AC$.

COROLLARY. Of all triangles having the same area, that which is equilateral has the least perimeter.

For the triangles having a given area and least perimeter must be isosceles, whichever side is taken for the base.

PROPOSITION II. *Of all triangles having the same base and equal perimeters, that which is isosceles has the greatest area.*

For, referring to the figure and proof of Proposition I. we see, that when A' is on the line AM the perimeter of $A'BC$ is greater than that of ABC; and if A' was still further from BC the perimeter of $A'BC$ would exceed that of $A'BC$ still more.

Hence A' must be between AM and BC and therefore the altitude of $A'BC$, would be less than that of ABC, and the area less also.

COROLLARY. Of all triangles having equal perimeters, that which is equilateral has the greatest area. For the triangle having a given perimeter and greatest area, must be isosceles, whichever side is taken for the base.

(*To be continued.*)

ARITHMETIC.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

28. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

A rectangular field (not a square one) contains as many acres as there are boards in the fence enclosing it. The fence is four boards high and each board is eleven feet long. How many acres in the field?

Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland; G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia, and the PROPOSER.

Let x and px feet represent the length and breadth,

$$\text{then } \frac{8(1+p)x}{11}=\frac{px^2}{9\times30\frac{1}{4}\times160}\ldots.(1).$$

$$\therefore \quad x=31680\left(1+\frac{1}{p}\right), \text{ and } px=31680(1+p).$$

Hence the number of boards in the fence is

$$B=\frac{8(1+p)x}{11},=\frac{23040(1+p)^2}{p}\ldots.(2);$$

and the number of acres in the field is

$$A=\frac{px^2}{9\times30\frac{1}{4}\times160},=\frac{[31680(1+p)]^2}{43560p}=\frac{23040(1+p)^2}{p}\ldots.(3).$$

Since the field is not to be a square one, put $p=2$; then $x=47520$, $px=95040$, $B=103680$, and $A=103680$. By giving to p any other values, we easily deduce the dimensions of an indefinite number of rectangular fields answering the conditions of the problem.

Also solved by *Hon. JOSIAH H. DRUMMOND, JOHN M. LOWELL, C. D. SCHMITT and B. F. ROBINS.*

PROBLEMS.

35. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

It costs $C=\$22$ to paper a room $a=18$ feet long, $b=15$ feet wide, and $c=10$ feet high, with paper m/n th, $=\frac{3}{4}$, of a yard wide. Find the price of the paper per roll of $R=12$ linear yards.

36. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

I have three jars, A, B, and C, holding respectively $a=1$, $b=3$, and $c=5$ gallons. A is empty, B is full of water, and C is full of wine. I *fill* A from B; then I *fill* up B from C and pour the contents of A into C. After *repeating* this operation, how much wine is there in B? How much in C?

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

24. Proposed by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Find x and y from the equations,

$$x^2+2xy^2-2xy=4y^3,$$
$$y^4+2x^3y^2-2xy^2=4x^4.$$

I. Solution by J. K. ELLWOOD, M. A., Principal of Colfax School, Pittsburg, Pennsylvania; and B. F. BURLESON, Oneida Castle, New York.

The first equation may be written $x^2+2x(y^2-y)=4y^3$.

Completing squares, $x^2+2x(y^2-y)+(y^2-y)^2=4y^3+(y^2-y)^2=(y^2+y)^2$. Whence $x+(y^2-y)=\pm(y^2+y)$, and $x=2y$ or $-2y^2$ ....(1).

The second equation may be written $y^4+2y^2(x^3-x)=4x^4$. Completing squares, $y^4+2y^2(x^3-x)+(x^3-x)^2=4x^4+(x^3-x)^2=(x^3-x)^2$,

whence $y^2=2x$ or $-2x^3$ ....(2).

Equating *positive* values of x from (1) and (2), we find $y=4$. Then $x=8$.

Solved with similar results by *J. H. DRUMMOND, P. S. BERG and the PROPOSER.*

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D.C., D. G. DORRANCE, Jr., Camden, New Jersey, and J. A. TIMMONS, A. M., Professor of Mathematics in St. Mary's College, St. Mary's, Kentucky.

Equation (2) becomes $y^4+2xy^2(x^2-1)=4x^4$.

Completing square and reducing, we get $y^2=2x$ or $-2x^3$;

$$\text{whence } x=\frac{y^2}{2} \text{ or} \left(-\frac{y^2}{2}\right)^{\frac{1}{3}}.$$

Equating these values of x with those found from Eq. (1), we have

$$2y=\frac{y^2}{2},\ -2y^2=\left(-\frac{y^2}{2}\right)^{\frac{1}{3}},\text{ and } 2y=\left(-\frac{y^2}{2}\right)^{\frac{1}{3}};$$

whence $y=0,\ 4, -\frac{1}{16}, \pm\frac{1}{2}$, and $x=0,\ 8, -\frac{1}{8}, -\frac{1}{2}$.

Solved with same results by *OTTO GECKELER, F. P. MATZ, COOPER D. SCHMITT, and G B.M.ZERR.*

III. Solution by C. E. WHITE, Trafalgar, Indiana; C. W. M. BLACK, A. M., Department of Mathematics, Wilmington Conference Academy, Dover, Delaware; and JOHN B. FAUGHT, Bloomington, Indiana.

By factoring the equations, they may be written

$$(x+2y^2)(x-2y)=0, \text{ and } (y^2+2x^3)(y^2-2x)=0.$$

$$\text{Whence } \begin{cases} x+2y^2=0\ldots.(1) \\ x-2y=0\ldots..(2) \end{cases}$$
$$\begin{cases} y^2+2x^3=0\ldots.(3) \\ y^2-2x=0\ldots.(4). \end{cases}$$

The values that will satisfy (1) and (3) are easily found to be $x=0,\ \pm\frac{1}{2}$ and $y=0,\ \pm\frac{1}{2}\sqrt{(-1)},\ \pm\frac{1}{2}$; (1) and (4), $x=0$, and $y=0$; (2) and (3), $x=0, -\frac{1}{8}$, and $y=0, -\frac{1}{16}$; (2) and (4), $x=0,\ 8$, and $y=0,\ 4$.

$\therefore\quad x=0,\ 8, -\frac{1}{8}, \pm\frac{1}{2}$, and $y=0,\ 4,\ -\frac{1}{16}, \pm\frac{1}{2}, \pm\frac{1}{2}\sqrt{(-1)}$.

Solved with like results by *J. F. W. SCHEFFER.* Also solved by *J. A. CALDERHEAD, H. W. DRAUGHON, A. L. FOOTE, J. H. GROVES, and H. C. WHITAKER.*

28.* Proposed by P. C. CULLEN, Meade, Nebraska.

A man and a boy get n dollars for digging potatoes, the man can dig them as fast as the boy can pull the vines, but the man can pull vines m times as fast as the boy can dig them. Divide the money.

I. Solution by A. L. FOOTE, C. E., Merrick, New York; P. H. PHILBRICK, C. E., Lake Charles, Louisiana; and J. A. CALDERHEAD, B. Sc., Limaville, Ohio.

Let x be the cost of all the digging, and y the cost of pulling the vines; then $x+y=n \dots (1)$. Now by the conditions the man can perform the effect x, while the boy performs the effect y. Also the man can do my, while the boy does x. Therefore we have $x : my :: y : x$, or $x^2=my^2 \quad \dots (2)$.
$\therefore\quad x=y\sqrt{(m)} \dots (3)$. Substituting in (1) we have

$$y=\frac{n}{1+\sqrt{(m)}},\ \text{and}\ x=\frac{n\sqrt{(m)}}{1+\sqrt{(m)}}.$$

II. Solution by J. H. DRUMMOND, LL. D., Portland, Maine.

The problem is indeterminate unless we assume the man's rate of digging and of pulling vines bears the same ratio to each other as the boys. Let x = the man's time for pulling vines and px, his time of digging; and y = boy's time for pulling vines and py his time of digging. Then $\frac{1}{x}$ and $\frac{1}{px}$ = man's rate of digging and pulling vines, respectively, and $\frac{1}{y}$ and $\frac{1}{py}$, boy's rate.

Then $\frac{1}{px}=\frac{1}{y}$, and $\frac{1}{x}=\frac{m}{py}$. Hence $y=px$ and $y=\frac{mx}{p}=px$. Hence $p^2=m$, and $p=\sqrt{m}$

$\therefore\ y=x\sqrt{m}$. $\frac{1}{x}+\frac{1}{px}$ = man's work in same time boy works, $\frac{1}{y}+\frac{1}{py}$, or $\frac{1}{x\sqrt{m}}+\frac{1}{mx}$. Then $\frac{1}{x}+\frac{1}{x\sqrt{m}}+\frac{1}{x\sqrt{m}}+\frac{1}{mx}$ = what both would do in same time, or $\frac{m+2\sqrt{m}+1}{mx}$.

$$\text{Then } n\left(\frac{m+\sqrt{m}}{mx}\right)\div\frac{m+2\sqrt{m}+1}{mx}=\text{man's part}=\frac{n\sqrt{m}}{1+\sqrt{m}},\ \text{and}$$

$$n\left(\frac{\sqrt{m}+1}{mx}\right)\div\frac{(1+\sqrt{m})^2}{mx}=\text{boy's part}=\frac{n}{1+\sqrt{m}}.$$

III. Solution by J. H. GROVE, Howard College, Brownwood, Texas.

Let d = the amount of work digging a certain distance, and p = the

amount of work pulling vines the same distance. Also let x=the amount of money the man received and, y=the amount of money the boy received. Since the amounts of money are to each other as the amounts of work done.

$\therefore x:y :: d:p$; but $x:y :: mp:d$.

$\therefore mp:d :: d:p :: x:y. \quad \therefore m=\dfrac{d^2}{p^2}=\dfrac{x^2}{y^2}$.

$\therefore x=y\sqrt{m}$. But since $x+y=n$. $\therefore x=n-y$.

$\therefore y\sqrt{m}=n-y. \quad \therefore y=\dfrac{n}{1+\sqrt{m}}$, and $x=\dfrac{n\sqrt{m}}{1+\sqrt{m}}$.

Also solved by *P. S. BERG, C. W. M. BLACK, J. K. ELLWOOD, M. A GRUBER, F. P. MATZ, C. E. WHITE, J. F. W SCHEFFER, H. C. WHITAKER*, and *G. B. M. ZERR.*

26. Proposed by ALVIN E. SCHMIDT, Winesberg, Ohio.

Show that $abc>(a+b-c)(x+c-b)(b+c-a)$ unless $a=b=c$.

I Solution by P. S. BERG, Apple Creek, Ohio.

$$a^2>a^2-(b-c)^2$$
$$b^2>b^2-(a-c)^2$$
$$c^2>c^2-(a-b)^2$$

Multiplying together the corresponding members of these inequalities, $a^2b^2c^2>(a+b-c)^2(a+c-b)^2(b+c-a)^2$.

$\therefore abc>(a+b-c)(a+b-c)(b+c-a$.

II. Solution by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

Such examples can be proved either by beginning with known principles and ending with the example, or beginning with the example and reducing it to known truths. I will use the latter method.

If $abc>(b+c-a)(a+b-c)(c+a-b)$, then $abc>ab(a+b)+bc(a+c)+ac(a+c)-2abc-a^3-b^3-c^3$ by multiplication, or $a^3+b^3+c^3+3abc>ab(a+b)+bc(b+c)+ac(a+c)$, but $a^3+b^3+c^3>3abc$. Hall and Knight's Algebra, or easily proved. Hence, *a fortiori*, $2(a^3+b^3c+{}^3)>ab(a+b)+bc(b+c)+ac(a+c)$, but this is true, Hall and Knight's Algebra, page 210.

Hence, the original proposition is true.

Also elegantly solved by *B. F. BURLESON, F. P. MATZ*, and *G. B. M. ZERR.*

PROBLEMS.

36. Proposed by J. A. CALDERHEAD, B. Sc., Superintendent of Schools, Limaville, Ohio.

Resolve $(x^2+y^2)(x^2+z^2)(y^2+z)$ into the sum of two squares.

37. Proposed by H. M. CASH, Gibson, Ohio.

The area of the segment of a circle=c, and radius=r. Find height of segment.

38 Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A man sold 2 horses and a mule for $286.90. On the first horse he gained as much per cent. as the horse cost dollars, and gained $\frac{5}{8}$ as much per cent. on the second horse as the first, and he loses $9.10 on the mule. His net gain was $86.90. What was the cost and selling price of each?

GEOMETRY.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

23. Proposed by E. L. PRATT, Tecumseh, Nebraska.

The ordinate of the point P of an ellipse is produced to meet the circle described on the major axis as diameter at Q. CQ, the straight line joining Q and the center of the ellipse, is tangent to the circle described on the focal radius of P as diameter.

If θ is the excentric angle prove that

$$\sin 2\theta=\frac{2(2a+b)\pm4\sqrt{a(a+b)}}{a-b}.$$

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

I get a different result from that given in the problem by the following method:

Let (x', y') be the coordinates of P; then the focal distance of P is $a-ex'$, and the half of it is the radius of the circle. The coordinates of the center of the circle are plainly $\frac{1}{2}(x'+ae)$, $\frac{y'}{2}$, and the equation to the circle is $\left(x-\frac{x'+ae}{2}\right)^2+\left(y-\frac{y'}{2}\right)^2=\left(\frac{a-ex'}{2}\right)^2$....(1), or, reducing, $x^2+y^2-(x'+ae)x-y'y+aex'=0$....(2).

The coordinates of Q are $\left(x', \frac{a}{b}y'\right)$ and the equation to CQ is $ay'x-bx'y=0$....(3).

The condition that (3) touches (2) is given by $a^2y'^2\left(aex'-\frac{y'^2}{4}\right)$ $+b^2x'^2\left(aex'-\frac{(x'+ae)^2}{4}\right)-\frac{1}{2}abx'y'^2(x'+ae)=0$....(4), or $4a^3b^2ex'$ $=[ay'^2+bx'(x'+ae)]^2$....(5).

But $x'=a\cos\theta$, and $y'=b\sin\theta$; and (5) becomes $4a^4b^2e\cos\theta$ $=[ab^2\sin^2\theta+ab\cos\theta(a\cos\theta+ae)]^2$ or $4a^2e\cos\theta$ $=[b\sin^2\theta+a\cos^2\theta+ae\cos\theta]^2$....(6), a biquadratic in $\sin\theta$ or $\cos\theta$, consistant with the fact that there are four positions of P possible.

II. Solution by the PROPOSER.

Let (x', y') be the coordinates of the point P, and $\left(\frac{x_1+e}{2}, \frac{y}{2}\right)$ the coordinates of the middle point of FP, the focal radius, where e is the linear excentricity of the ellipse. The equation of CQ is $y=x\tan\theta$. Employing the

usual formulas, $FP=\frac{a^2-ex_1}{a}$; and the lengths of the perpendicular from the point $\left(\frac{x_1+e}{2}, \frac{y_1}{2}\right)$ upon CQ is $\frac{y_1-(x_1+e)\tan\theta}{\pm 2\sqrt{1+\tan\theta}}$.

Since CQ is tangent to the circle upon FP this perpendicular$=\frac{FP}{2}$

Therefore, substituting for (x_1, y_1) their values in terms of θ, we obtain,

$$(a-b)\sin\theta\cos\theta-a=-e(\sin\theta+\cos\theta).$$

Squaring and arranging $(a-b)^2\sin^2\theta\cos^2\theta-2(a-b)(2a+b)\sin\theta\cos\theta=-b^2$, which may be written $(a-b)^2\sin^2 2\theta-4(a-b)(2a+b)\sin 2\theta=-4b^2$.

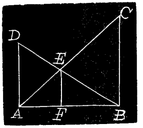
Solving, $\sin 2\theta=\frac{2(2a+b)\pm4\sqrt{[a(a+b)]}}{a-b}$.

Excellent solutions were also received from *J. F. W. SCHEFFER*, *G. B. M. ZERR*, and *JOHN FAUGHT*.

24. Proposed by T. W. PALMER, Professor of Mathematics in the University of Alabama.

Two right triangles have the same base, the hypotenuses of the first is equal to 60, of the second 40. The point of intersection of the two hypotenuses is at the distance 15 from the base. Find the length of the base.

Solution by SETH PRATT, C. E, Assyria, Michigan; A. H. BELL, Hillsboro, Illinois; and P. S. BERG, Apple Creek, Ohio.



Let $AD=x-y$; $BC=x+y$; $EF=c=15$; $BD=b=40$; $AC=a=60$; $AF=z$; and $FB=w$. By similar triangles, $w:c=w+z:x-y$, or $w(x-y)=c(w+z)\ldots(1)$; and $z:c=w+z:x+y$, or $(w+z)c=z(x+y)\ldots(2)$.

$\therefore w(x-y)=z(x+y)\quad\ldots(3)$. Put $x+y=s$ and $x-y=t$. Then $wt=zs$ or $w=zs/t\ldots(4)$. Substituting this value of w in the proportion of (2), $\frac{2s}{t}:c=\frac{zt+zs}{t}:t$. Dividing antecedents by $\frac{z}{t}$, we have $s:c=t+s:t$, or $st=c(t+s)$, whence $c=\frac{st}{s+t}$ * $=\frac{(x+y)(x-y)}{x+y+x-y}=\frac{x^2-y^2}{2x}\ldots(5)$.

From (5), $x^2-2cx=y^2$, whence $x=c+\sqrt{(y^2+c^2)}$. From the figure, $AC^2-BC^2=AB^2$ and $BD^2-AD^2=AB^2$. $\therefore AC^2-BC^2=BD^2-AD^2$, or $a^2-(x+y)^2=b^2-(x-y)^2$, whence $x=\frac{a^2-b^2}{4y}$. Equating the two values of x, $\frac{a^2-b^2}{4y}=c+\sqrt{(y^2+c^2)}$.

From this, we have $y^4+\frac{1}{2}c(a^2-b^2)y=(a^2-b^2)^2$.

Restoring numbers, and solving the resulting equation, $y=14.060811$.

*The sequence is, that AB may be any number whatever when AC and BD are indefinite.

$x=\frac{a^2-b^2}{y}=\frac{500}{y}=35.559825.$ $x+y=49.620636$, $x-y=21.499014$.

$AB=\sqrt{(AC^2-BC^2)}=\sqrt{[a^2-(x+y)^2]}=33.731178$.

Solutions were also received from *G. B. M. ZERR, J. F. W. SCHEFFER, L. B. FRAKER*, and *J. K. ELLWOOD*.

25. Proposed by L. B. FRAKER, Weston, Ohio.

The sides of a quadrilateral board are $AB=7$ inches, $BC=15$ inches, $CD=21$ inches, and $DA=13$ inches; radius of incribed circle is 6 inches. (1) What are dimensions of the largest rectangular board that can be cut out of the given board, (2) largest square, (3) largest equilateral triangle? (Please solve without use of the calculus.)

Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let $Aa=Ad=x$, $Ba=Bb=y$, $Cb=Cc=z$, $Dc=Dd=w$,

then $x+y=AB=7=6\cot\frac{1}{2}A+6\cot\frac{1}{2}B\ldots.(1)$,

$y+z=BC=15=6\cot\frac{1}{2}B+6\cot\frac{1}{2}C\ldots.(2)$,

$z+w=CD=21=6\cot\frac{1}{2}C+6\cot\frac{1}{2}D\ldots.(3)$,

$w+x=DA=13=6\cot\frac{1}{2}D+6\cot\frac{1}{2}A\ldots.(4)$.

$\therefore\ 28=6\cot\frac{1}{2}A+6\cot\frac{1}{2}B+6\cot\frac{1}{2}C+6\cot\frac{1}{2}D\ldots.(5)$.

$(5)-\{(1)+(4)\}$ gives $4=3\cot\frac{1}{2}C-3\cot\frac{1}{2}A$

$\therefore\ 4\sin\frac{1}{2}A\sin\frac{1}{2}C=3\sin\frac{1}{2}A\cos\frac{1}{2}C-3\cos\frac{1}{2}A\sin\frac{1}{2}C\ldots.(6)$.

$(5)-\{(3)+(4)\}$ gives $1=\cos\frac{1}{2}D-\cos\frac{1}{2}B$

$\therefore\ \sin\frac{1}{2}D\sin\frac{1}{2}B=\sin\frac{1}{2}B\cos\frac{1}{2}D-\cos\frac{1}{2}B\sin\frac{1}{2}D\ldots.(7)$.

From DB^2 we get $45\cos C-13\cos A=32$.

$\therefore\ 13\sin^2\frac{1}{2}A=45\sin^2\frac{1}{2}C\ldots.(8)$.

$45\cos^2\frac{1}{2}C-13\cos^2\frac{1}{2}A=32\ldots.(9)$.

From AC^2 we get $13\cos D-5\cos B=8$.

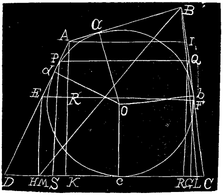
$\therefore\ 13\sin^2\frac{1}{2}D=5\sin^2\frac{1}{2}B\ldots.(10)$.

$13\cos^2\frac{1}{2}D-5\cos^2\frac{1}{2}B=8\ldots.(11)$.

From (6), (7), (8), (9), (10), (11) we get

$A=134°\ 45'\ 35''$, $B=106°\ 15'\ 39''$, $C=D=59°\ 29'\ 23''$.

Now the greatest equilateral triangle is the one which has the greatest side. Since $D=C$ only two cases confront us, the one is to draw a line from B making an angle of 60° with DC, the other to draw a line from some point in DC making an angle of 6° with BC. The former line is found to be very little the longer. Let BL be this line, then we have $15 : BL=\sin 120° : \sin 59°\ 29'\ 23''$.



$\therefore\ BL=$side of triangle$=14.92$ inches. $\therefore\ BLM=$triangle. Also since $D=C$, the greatest square has its side coincident with DC. Hence $\frac{21-x}{2} : x=\cos 59°\ 29'\ 23'' : \sin 59°\ 29'\ 23''$.

$\therefore\ x=$side of square$=9.64$ inches, and $PQRS=$square. Also since $D=C$, the side of the greatest rectangle will

coincide with DC. Draw AI parallel and AK perpendicular to DC and let $EFGH$ be the rectangle.

Then $\frac{1}{2}(AI+DC)\times AK=EF\times FG+FG\times GC+\frac{1}{2}(AI+EF)(AK-KR')$.
But $AK=11.2$ inches, $AI=7.8$ inches. $\therefore$ $588=28\,EF+33FG$.

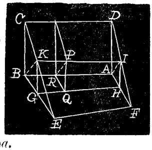
$\therefore$ for maximum $28\,EF=33\,FG$. $\therefore$ $EF=10.5$ inches, FG 8.91 inches.

26. Proposed by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

$ABCD$ represents a rectangle, and $ABEF$ a trapezoid which is perpendicular to the rectangle, both figures having the side AB common to each other, and ADF and BCE forming two right triangles perpendicular to the rectangle $ABCD$. To determine the conoidal surface $CDFE$ so as to satisfy the condition that any plane laid through AB will interest it in a straight line. Also find volume of the surface thus formed.

Solution by the PROPOSER.

Let $BC=AD=h$, $AB=a$, $BE=b$, $AF=c$. Let P represent a point in the surface, and put $AR=x$, $RQ=y$, $PQ=z$



The triangles BGK, PQR, and AHI are similar, and we may now put $AH=ny$, $IH=nz$, $BG=my$, $KG=mz$; but $h:mz=b-my$; $h:nz=c:c-nz$, whence $m=\dfrac{bh}{hy+bz}$, $n=\dfrac{ch}{hy+cz}$.

In the trapezoid $AHGB$, we now have $AB=a$, $AH=\dfrac{chy}{hy+cz}$, $BG=\dfrac{bhy}{hy+bz}$, $AR=x$, $RQ=y$. $\therefore$ $(AH+y)x+(BG+y)(a-x)=(AH+BG)a$.

Substituting, clearing of fractions, and arranging, we find for the equation of the surface

$$abcz^2+a(b+c)hyz-(b-c)h^2xy+ah^2y^2-abchz-ach^2y=0.$$

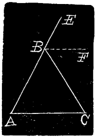
Let us now denote $\angle CBK=\angle DAI$ by θ, and angles BCK and ADI represent by C and D. For the volume we have $\frac{1}{6}ah^2\int_0^{\frac{\pi}{2}}\Big[\dfrac{\sin^2 C}{\sin^2(C+\theta)}$
$+\dfrac{\sin^2 D}{\sin^2(D+\theta)}+\dfrac{\sin C\sin D}{\sin(C+\theta)\sin(D+\theta)}\Big]d\theta=\frac{1}{6}ah^2\Big[\tan C+\tan D$
$+\dfrac{\tan C\tan D}{\tan C-\tan D}\log\dfrac{\tan C}{\tan D}\Big]$ but $\tan C=\dfrac{b}{h}$, $\tan D=\dfrac{c}{h}$;

$$\therefore \text{volume}=\tfrac{1}{6}ah\Big[b+c+\frac{bc}{b-c}\log\frac{b}{c}\Big].$$

27. Proposed by ADOLPH BAILOFF, Durand Wisconsin.

A line BF, that bisects an angle exterior to the vertical angle of an isosceles triangle is parallel to the base AC.

Solution by Mrs. MARY E. HOGSETT, Danville, Kentucky; P. S. BERG, Apple Creek, Ohio, Professors JOHN FAUGHT, Bloomington, Indiana; and M. A. GRUBER, War Department, Washington, D. C.



$\angle EBC = \angle A + \angle C = 2\angle C$.

But $\angle EBC = 2\angle FBC$, since BF is the bisector of $\angle EBC$. $\therefore 2\angle FBC = 2\angle C$, or $\angle FBC = \angle C$.

Hence, BF is parallel to AC, because if two straight lnesi are cut by a third straight making the alternate interior angles equal, the two lines are parallel.

Solutions were also received from *J. K. ELLWOOD, H. G. WHITAKER*, and *G. B. M. JERR*.

NOTE—No solution has yet been received to problem 20.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

20. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

$$\int_0^{\frac{1}{2}\pi} \sqrt{[(1-e^2\cos^2\phi)(1-e^2\sin^2\phi)]}\,d\phi = \text{what?}$$

Solution by Professor J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

$$\sqrt{(1-e^2\cos^2\phi)(1-e^2\sin^2\phi)} = \sqrt{1-e^2(\cos^2\phi+\sin^2\phi)+e^4\sin^2\phi\cos^2\phi}$$

$$=\sqrt{1-e^2+\frac{e^4}{4}\sin^2 2\phi} = \sqrt{1-e^2+\frac{e^2}{4}-\frac{e^4}{4}\cos^2 2\phi}$$

$$=\tfrac{1}{2}\sqrt{(2-e^2)^2-e^4\cos^2 2\phi} = \tfrac{1}{2}\sqrt{(2-e^2)^2-e^4\sin^2\left(\frac{\pi}{2}-2\phi\right)}$$

$$=\tfrac{1}{2}(2-e^2)^2\sqrt{1-\frac{e^4}{(2-e^2)^2}\sin^2\left(\frac{\pi}{2}-2\phi\right)}.$$

$$\therefore \int_0^{\frac{\pi}{2}} d\phi\sqrt{(1-e^2\cos^2\phi)(1-e^2\sin^2\phi)}$$

$$=-\tfrac{1}{4}(2-e^2)\int_{\frac{1}{2}\pi}^{-\frac{1}{2}\pi} d\left(\frac{\pi}{2}-2\phi\right)\sqrt{1-\frac{e^4}{(2-e^2)^2}\sin^2\left(\frac{\pi}{2}-2\phi\right)}$$

$$=\tfrac{1}{4}(2-e^2)\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} d\lambda\sqrt{1-\frac{e^4}{(2-e^2)^2}\sin^2\lambda}$$

$$=\tfrac{1}{2}(2-e^2)\int_0^{\frac{1}{2}\pi}dx\sqrt{1-\frac{e^4}{(2-e^2)^2}\sin^2\lambda}=\tfrac{1}{2}(2-e^2)E\left(\frac{e^2}{2-e^2},\ \frac{\pi}{2}\right)=\tfrac{1}{2},\text{ if } e=1.$$

Also solved by *OTTO GECKELER, WILLIAM HOOVER, ARTEMAS MARTIN, F. P. MATZ, and G. B. M. ZERR.*

21. Proposed by T. JOHN COLE, Columbus, Ohio.

In the equilateral triangle ABC, AB the base is 10 feet. With B as a center an arc is drawn from C to A; likewise with A as a center an arc is drawn from C to B. What is the volume of the solid generated by revolving the figure about the altitude of the triangle as an axis.

I. Solution by H. C. WHITAKER, B. S., C. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

The triangle revolving describes a cone the volume of which is $\frac{1}{3}\pi.25.5\surd3=226.725$.

The area of either segment is $50\left(\frac{\pi}{3}-\frac{\surd3}{2}\right)=9.05861$; the distance of the center of gravity of the segment from the center of the circle $=\frac{1000}{12\text{ Area}}$, and from the axis of revolution is $\left(\frac{1000}{12\text{ Area}}-5\sec30^\circ\right)\cos30^\circ=2.9639$.

The product of the area by the path of the center of gravity $9.05861\times 2.\pi.2.9569=168.865$.

Adding to this the volume of the cone, the answer is 395.59 cubic feet.

II. Solution by JOHN DOLMAN, Jr., Counsellor at Law, Philadelphia, Pennsylvania.

Let the origin be at the centre of the base. Put the side of the triangle$=2r$, and let y=the radius of any circular horizontal section at any variable height x.

Then $V=\int_0^{r\surd3}\pi y^2dx\ldots.(1)$. The equation of the arc forming the side is $4r^2=x^2+(y+r)^2$ from which $y^2=5r^2-x^2-2r\surd(4r^2-x^2)$ which value of y^2 substituted in (1) gives $V=\pi\int_0^{r\surd3}[(5r^2-x^2-2r\surd(4r^2-x^2)]dx$

$$=\pi\left[5r^2x-\tfrac{1}{3}x^3-r[x\surd(4r^2-x^2)+4r^2\sin^{-1}\frac{x}{2r}\right]_0^{r\surd3}$$

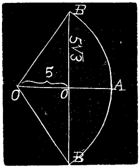
$=\pi(3r^3\surd3-\frac{4}{3}\ \pi r^3)=\frac{\pi r^3}{3}(9\surd3-4\pi)=3.164722r^3$, and putting $r=5$ this gives $V=395.59+$cubic feet.

III. Solution by OTTO GECKELER, Bloomington, Indiana.

The equation of circle with O as origin is $y^2+(x-5)^2=100$, or $x=5$

$-\sqrt{(100-y^2)}$ on part of curve in question. Then from conditions of problem

$$V=2\pi\int_0^{5\sqrt{3}}\int_0^{5-\sqrt{(100-y^2)}}xdxdy=\pi\int_0^{5\sqrt{3}}[125-y^2-10\sqrt{(a^2-y^2)}]dy.$$

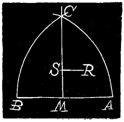


$$=\pi\left[125\times5\sqrt{3}-125\sqrt{3}-1000\int_0^{\theta=\sin^{-1}\frac{1}{2}\sqrt{3}}\cos^2\theta d\theta\right]$$

$$=\pi\left[125\times4\times\sqrt{3}-1000\left(\frac{\sqrt{3}}{8}+\frac{\pi}{6}\right)\right]=\pi\left[375\sqrt{3}-\frac{1000}{6}\pi\right]=\frac{125\pi}{3}(9\sqrt{3}-4\pi)=395.59+\text{cubic feet.}$$

IV. **Solution by W. WIGGINS, Richmond, Indiana.**

The volume generated is equal to that of cylinder of base CMA, and altitude equal to the length of the path scolpt out by R, the center of gravity of the area CMA. (Guldins theorem)



Taking B as the origin of co-ordinates and BA as the axis of x, we have for the area of AMC,

$$\text{Area}=\tfrac{1}{2}.100.\tfrac{\pi}{3}-\tfrac{1}{2}.100.\tfrac{1}{2}.\tfrac{1}{2}\sqrt{3}=\tfrac{1}{6}(100\pi-75\sqrt{3}).$$

If x', y' denote the co-ordinates of c, g, of the area CMA, x'. area $CMA=\int_0^{10}x\sqrt{(100-x^2)}dx$.

x'. area$=\frac{1}{3}(100-x^2)^{\frac{3}{2}}$ x'. area$=\frac{1}{3}y^3$, where $y=5\sqrt{3}$.

$\therefore\ x'=\dfrac{[5\sqrt{3}]^3}{3\times\text{Area}}$, or $x'=\dfrac{125\sqrt{3}}{\text{Area}}$, $\therefore\ SR=\dfrac{125\sqrt{3}}{\text{Area}}-5$. Therefore if h denote the altitude of the equivalent cylinder, we have $h=2\pi\left(\dfrac{125\sqrt{3}}{\text{Area}}-5\right)$.

$$V=2\pi\left(\frac{125\sqrt{3}}{\text{Area}}-5\right)\times\text{Area}=2\pi(125\sqrt{3}-5\text{ Area})=2\pi[125\sqrt{3}$$

$$-\tfrac{5}{6}(100\pi-75\sqrt{3})]=\frac{125\pi}{3}(9\sqrt{3}-4\pi)=395.59+\text{ cubic feet.}$$

Also variously solved by *C. W. M. BLACK, A. L. FOOTE, J. F. W. SCHEFFER, G. B. M. ZERR, J. H. DRUMMOND, F. P. MATZ, and P. H. PHILBRICK*.

PROBLEMS.

28. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

How far from the stage must Miss Love sit in order that she *may see to best advantage* Mr. Rich deliver the valedictory oration?

29. **Proposed by CHARLES E. MYERS, Canton, Ohio.**

A hen running at the rate of $n=2$ feet per second, on the circumference of a circle, radius $r=50$ feet, is observed by a hawk $a=600$ feet directly above the center.

The hawk at once starts in pursuit, flying at the rate of $m=5$ feet per second and keeping always in a straight line with the starting point and the hen.

Determine the path followed and the distance the hawk will fly before catching the hen.

MECHANICS.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

11. Proposed by CHARLES E. MYERS, Canton, Ohio.

"A homogeneous sphere moves down a rough inclined plane, whose angle of inclination θ to the horizon is greater than that of the angle of friction: if the coefficient of friction is less than $\frac{2}{7}$ tang. θ, show that the sphere will roll and slide down the inclined plane."

Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let a=the radius of the sphere, ϕ the angle turned through by the sphere.

Resolving along the perpendicular to the inclined plane we have, if mk^2 be the moment of inertia of the sphere about a horizontal diameter, $m\frac{d^2x}{dt^2}=mg\sin\theta-F\ldots(1)$, where F is friction acting along the plane at the point of contact of the sphere and mg acting vertically at the centre.

Also $m\frac{d^2y}{dt^2}=-mg\cos\theta+R\ldots(2)$, where R is the reaction perpendicular to the plane. In order to avoid reactions let us take moments about the point of contact, and we get $ma\frac{d^2x}{dt^2}+mk^2\frac{d^2\phi}{dt^2}=mga\sin\theta\ldots(3)$.

Since there is no jump, $y=a\ldots(4)$.

From (1) $F=\frac{2}{7}mg\sin\theta\ldots(5)$, from (2) and (4) $R=mg\cos\theta\ldots(6)$.

$\therefore$ from (5) and (6) $F=\frac{2}{7}R\tan\theta$ but since μ=coefficient of friction $<\frac{2}{7}\tan\theta$, $F=\mu R$ and the equations of motion become $m\frac{d^2x}{dt^2}=mg\sin\theta-\mu R\ldots(7)$, $0=-mg\cos\theta+R\ldots(8)$. $ma\frac{d^2x}{dt^2}+mk^2\frac{d^2\phi}{dt^2}=mga\sin\theta\ldots(9)$.

From (8) $R=mg\cos\theta$, this in (7) gives $\frac{d^2x}{dt^2}=g(\sin\theta-\mu\cos\theta)$,

$\therefore$ $x=\frac{1}{2}gt^2(\sin\theta-\mu\cos\theta)$ since the sphere starts from rest. Also $k^2=\frac{2}{5}a^2$.

$\therefore$ (9) becomes $\dfrac{d^2x}{dt^2}+\frac{2}{5}a\dfrac{d^2\phi}{dt^2}=g\sin\theta$. Substituting $\dfrac{d^2x}{dt^2}$ we get $g\sin\theta-g\mu\cos\theta+\frac{2}{5}a\dfrac{d^2\phi}{dt^2}=g\sin\theta$.

$\therefore a\dfrac{d^2\phi}{dt^2}=\frac{5}{2}\mu g\cos\theta$. $\therefore \phi=\frac{5}{4}\mu\dfrac{g}{a}t^2\cos\theta$. Also $\dfrac{dx}{dt}$

$$=gt(\sin\theta-\mu\cos\theta),\ a\frac{d\phi}{dt}=\tfrac{5}{2}\mu gt\cos\theta.$$

$\therefore \dfrac{dx}{dt}-a\dfrac{d\phi}{dt}=gt(\sin\theta-\frac{7}{2}\mu\cos\theta)$ is the velocity of the point of the sphere in contact with the plane. Since $\mu<\frac{2}{7}\tan\theta$, this velocity can never vanish. $\therefore$ the friction will never change to rolling friction. This completely determines the motion.

II. Solution by WILLIAM HOOVER, A. M., Ph. D., Athens, Ohio.

Let a=the radius of the sphere, $k^2=\frac{2}{5}a^2$=the square of the radius of gyration about its center, μ=the coefficient of friction, s=the distance passed over by the center in the time t from the beginning of motion, perfect rolling being assumed, R=the normal reaction of the plane, F=the friction, ϕ=the angular rotation of the sphere, g=the acceleration of gravity, and m=the mass of the sphere.

Resolving parallel and perpendicular to the plane, and taking moments about the center of the sphere, $m\dfrac{d^2s}{dt^2}=mg\sin\theta-F\ldots.(1)$,

$$R=mg\cos\theta\ldots.(2)\text{ and }mk^2\frac{d^2\phi}{dt^2}=Fa\ldots.(3).$$

We have, also, $s=a\phi\ldots.(4)$, and then $\dfrac{ds}{dt}=a\dfrac{d\phi}{dt}\ldots.(5)$,

$$\frac{d^2s}{dt^2}=a\frac{d^2\phi}{dt^2}\ldots.(6).$$

Eliminating F from (1) and (3) by aid of (6) we find

$$\frac{d^2s}{dt^2}=\frac{a^2g\sin\theta}{a^2+k^2}\ldots.(7),\text{ and then }\frac{d^2\phi}{dt^2}=\frac{ag\sin\theta}{a^2+k^2}\ldots.(8).$$

Substituting in (3) we have $F=\dfrac{mgk^2\sin\theta}{a^2+k^2}\ldots.(9)$. Then $\mu=\dfrac{F}{R}$ $=\dfrac{k^2}{a^2+k^2}\tan\theta\ldots.(10)$.

Substituting $\mu R=\mu mg\cos\theta\ldots.(11)$ for F in (1) and (2), and integrating once, $\dfrac{ds}{dt}=gt(\sin\theta-\mu\cos\theta)+c\ldots.(12)$. $\dfrac{d\phi}{dt}=\dfrac{\mu agt}{k^2}\cos\theta+\phi'\ldots.(13)$

ϕ' being the initial angular velocity.

From (12) and (13) we have

$$\frac{ds}{dt}-a\frac{d\phi}{dt}=gt\cos\theta\,\frac{a^2+k^2}{k^2}\left(\frac{k^2}{a^2+k^2}\tan\theta-\mu\right)\ldots\ldots(14).$$

By hypothesis, $\mu<\frac{k^2}{a^2+k^2}\tan\theta$ or $<\frac{2}{7}\tan\theta$, and therefore $\frac{ds}{dt}-a\frac{d\phi}{dt}$ is positive, showing that all the motion is not rolling.

III. Solution by Professor P. H. PHILBRICK, M. S., C. E., Lake Charles, Louisiana.

Let a=the radius of the sphere, O the origin, A the point of the sphere in contact with the plane at O at starting and P the point in contact at the end of the time t. $OP=x$.

Let $ACP=\phi$, R=pressure on the plane, F=the sliding friction, and μ=coefficient of sliding friction$=\frac{F}{R}$.

Suppose the plane just rough enough to prevent sliding.

Then the equation for translation is $m\frac{d^2x}{dt^2}=mg\sin\theta-F\ldots(1)$,

and for rotation $mk_1^2\frac{d^2\phi}{dt^2}=aF\ldots(2)$, in which k_1=the radius of gyration and $\therefore$ $k_1^2=\frac{2}{5}a^2$.

Also $x=OP=arc\ AP=a\phi$; and differentiating twice gives, $\frac{d^2x}{dt^2}=a\frac{d^2\phi}{dt^2}\ldots(3)$.

Multiply (1) by a and add to (2) and get, $ma\frac{d^2x}{dt^2}+mk_1^2\frac{d^2\phi}{dt^2}=mg\sin\theta\ldots(4)$.

Substituting from (3) we get, $\frac{d^2\phi}{dt^2}=\frac{a}{a^2+k_1^2}g\sin\theta$.

From (2) and (5), $F=\frac{mk_1^2}{a}\cdot\frac{d^2\phi}{dt^2}=\frac{mk_1^2}{a}\cdot\frac{a}{a^2+k_1^2}g\sin\theta=\frac{mk_1^2}{a^2+k_1^2}g\sin\theta$.

Also $R=mg\cos\theta$. $\therefore$ $\mu=\frac{F}{R}=\frac{k_1^2}{k_1^2+a^2}\tan\theta=\frac{2}{7}\tan\theta$.

Hence if μ is less than $\frac{2}{7}\tan\theta$, it will all be utilized in causing rolling but it is insufficient to prevent some sliding.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

11. Proposed by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Find three whole numbers such that the square of the sum of any two of them diminished by the square of the other number shall be a square.

Solution by the PROPOSER.

Let rx, ry, and rz represent the required numbers.

Then we must have

$$(rx+ry)^2-r^2z^2=\square=r^2a^2\ldots.(1),$$
$$(rx+rz)^2-r^2y^2=\square=r^2b^2\ldots.(2),$$
$$(ry+rz)^2-r^2x^2=\square=r^2c^2\ldots.(3).$$

Expunging the square factor r^2 we have

$$(x+y)^2-z^2=a^2\ldots.(4),$$
$$(x+z)^2-y^2=b^2\ldots.(5),$$
$$(y+z)^2-x^2=c^2\ldots.(6).$$

Adding (4), (5), and (6) we get

$$(x+y+z)^2=a^2+b^2+c^2,\text{ or } x+y+z=\sqrt{(a^2+b^2+c^2)}\ldots.(7),$$

and we must find $a^2+b^2+c^2=\square$.

From (4), (5), (6) and (7) we readily find

$$x=\frac{a^2+b^2}{2\sqrt{(a^2+b^2+c^2)}},\ y=\frac{a^2+c^2}{2\sqrt{(a^2+b^2+c^2)}},\ z=\frac{b^2+c^2}{2\sqrt{(a^2+b^2+c^2)}}.$$

If now we take $r=2\sqrt{(a^2+b^2+c^2)}$ we have

$$rx=a^2+b^2,\ ry=a^2+c^2, rz=b^2+c^2,$$

subject to the condition $a^2+b^2+c^2=\square$. The solution of this condition can be effected in many ways. See *Mathematical Magazine*, Vol. II., No. 5, pp, 71-74. Using the simple formula near the bottom of page 71, viz: and putting $a=p$, $b=p+1$, $c=p^2+p$, we have

$$a^2+b^2+c^2=p^2+(p+1)^2+(p^2+p)^2-(p^2+p+1)^2,$$

where p may have any integral value that will make the numbers all different. Take $p=2$, then $a=2$, $b=3$, $c=6$, $rx=13$, $ry=40$, $rz=45$, the numbers sought.

II. Solution by H. W. DRAUGHON, Clinton, Louisiana.

Put $(m^2+n^2)-y=$1st number, $y=$2nd number, and $m^2-n^2=$3rd number. Squaring the sum of the 1st and 2nd, and diminshing by the square of the 3rd, we have $(m^2+n^2)^2-(m^2-n^2)^2=4m^2n^2$, a square.

We have therefore only to make,

$$[(m^2+n^2-y)+m^2-n^2]^2-y^2=4m^2(m^2-y^2)=\square\ldots.(1),\text{ and}$$
$$(m^2-n^2+y)^2-(m^2+n^2-y)^2=4m^2(y-n^2)=\square\ldots.(2),$$

in (1) make $m^2-y=p^2\ldots.(3)$ and in (2), make $y-n^2=q^2\ldots.(4)$.

From (3) and (4) we have, $n^2+q^2+p^2=m^2$.

This equation will be satisfied when, $n=(a^2+b^2-c^2)$, $q=2ac$, $p=2bc$, and $m=(a^2+b^2+c^2)$; a, b, and c can have any values that will not make any one of the required numbers greater than the sum of the other two.

Ex. Put $a=3$, $b=2$, and $c=1$, then $n=12$, $q=6$, $p=4$, and $m=14$.

From (3) or (4) we find $y=180$. Substituting these values in the original expressions, we have the following numbers, 180, 160, and 52. Dividing by 4, these numbers become 45, 40, and 13.

Also solved by *C. W. M. BLACK, M. A. GRUBER, F. P. MATZ, J. F. W. SCHEFFER, and G. B. M. ZERR*

12. **Proposed by H. W. DRAUGHON, Clinton, Louisiana.**

Find three numbers such that, the sum of their cubes may be a square, and the sum of their squares a cube.

Solution by ARTEMAS MARTIN, LL.D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Let ax, bx and cx denote the numbers; then $(a^3+b^3+c^3)x^3=\square\ldots\ldots(1)$, $(a^2+b^2+c^2)x^2=\text{cube},=x^3$ say....(2) and we have $x=a^2+b^2+c^2$.

Substituting in (1), after expunging x^2, $(a^2+b^2+c^2)(a^3+b^3+c^3)=\square\ldots\ldots(3)$.

Let $mv=a$, $nv=b$, $pv=c$; then (3) becomes

$(m^2+n^2+p^2)(m^3+n^3+p^3)v=\square,=v^2$ say, after rejecting v^4; whence

$$v=(m^2+n^2+p^2)(m^3+n^3+p^3).$$

$$\therefore\quad a=m(m^2+n^2+p^2)(m^3+n^3+p^3),$$
$$b=n(m^2+n^2+p^2)(m^3+n^3+p^3),$$
$$c=p(m^2+n^2+p^2)(m^3+n^3+p^3);$$

$$\text{and } x=a^2+b^2+c^2=(m^2+n^2+p^2)^3(m^3+n^3+p^3)^2.$$

$$\text{Hence } ax=m(m^2+n^2+p^2)^4(m^3+n^3+p^3)^3,$$
$$bx=n(m^2+n^2+p^2)^4(m^3+n^3+p^3)^3,$$
$$cx=p(m^2+n^2+p^2)^4(m^3+n^3+p^3)^3.$$

If $m=1$, $n=2$, $p=3$, the numbers are, after dividing out the 6th power factor 6^6, 38416, 76832 and 115248.

Also solved by *H. W. DRAUGHON, F. P. MATZ, and G. B. M. ZERR.*

PROBLEMS.

18. **Proposed by Professor G. B. M. ZERR, A. M., Principal of Schools, Staunton, Virginia.**

Decompose into the sum of two squares the number $17^3.73^5$.

19. **Proposed by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.**

Find three positive integer numbers whose sum is a cube, and, also, the sum of any two diminished by the third a cube.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

5. **Proposed by DE VOLSON WOOD, M. A., C. E., Professor of Mechanical Engineering, Stevens Institute of Technology, Hoboken, New Jersey.**

An actual case suggested the following:

An equal number of white and black balls of equal size are thrown into a rec-

angular box, what is the probability that there will be contiguous contact of white balls from one end of the box to the opposite end? As a special example, suppose there are 30 balls in the length of the box, 10 in the width, and 5 (or 10) layers deep.

Solution by Professor P. H. PHILBRICK, M Sc., C. E., Lake Charles, Louisiana.

The least number of balls forming a row is 30; and the greatest numl er is $30+9+4=43$.

Suppose the box to be divided into $30\times10\times5=1500$ equal cubes an end of the box being shown in the figure. Then, in general, a row may connect any cube at one end with any one of a determinate number of cubes at the other end of the box. Call the diameter of a ball unity then the latteral deviation of the rows may range from 0 to 9 and the vertical deviation from 0 to 4.

Let $m=$the number of rows in the width $=10$ and $n=$the number of rows in the depth$=5$. Then the following table shows the number of rows of given numbers of balls in a row so far as the beginning and the ending of the rows are concerned. The figures over the numbers of rows show the deviations of the rows latterally and vertically or vertically and latterally.

30 $mn \qquad = 50$

31 $2(m-1)n+2m(n-1)=90+80=\overset{1,0}{170} \qquad =170$

32 $2(m-2)n+2m(n-2)+4(m-1)(n-1)=\overset{2,0}{140}+\overset{1,1}{144} \qquad =284$

33 $2(m-3)n+2m(n-3)+4(m-2)(n-1)+4(m-1)(n-2)=\overset{3,0}{110}+\overset{2,1}{236} \qquad =346$

34 $2(m-4)n+2m(n-4)+4(m-3)(n-1)+4(m-1)(n-3)+4(m-2)(n-2)$
$=\overset{4,0}{80}+\overset{3,1}{184}+\overset{2,2}{96}=360$

35 $2(m-5)n+4(m-4)(n-1)+4(m-1)(n-4)+4(m-3)(n-2)+4(m-2)(n-3)$
$=\overset{5,0}{50}+\overset{4,1}{32}+\overset{3,2}{148}=330$

36 $2(m-6)n+4(m-5)(n-1)+4(m-4)(n-2)+4(m-2)(n-4)+4(m-3)(n-3)$
$=\overset{6,0}{40}+\overset{5,1}{80}+\overset{4,2}{104}+\overset{3,3}{56}=280$

37 $2(m-7)n+4(m-6)(n-1)+4(m-5)(n-2)+4(m-4)(n-3)+4(m-3)(n-4)$
$=\overset{7,0}{70}+\overset{6,1}{64}+\overset{5,2}{60}+\overset{4,3}{76}=230$

38 $2(m-8)n+4(m-7)(n-1)+4(m-6)(n-2)+4(m-5)(n-3)+4(m-4)(n-4)$
$=\overset{8,0}{20}+\overset{7,1}{48}+\overset{6,2}{48}+\overset{5,3}{40}+\overset{4,4}{24}=180$

39 $2(m-9)n+4(m-8)(n-1)+4(m-7)(n-2)+4(m-6)(n-3)+4(m-5)(n-4)$
$=\overset{9,0}{10}+\overset{8,1}{32}+\overset{7,2}{36}+\overset{6,3}{32}+\overset{5,4}{20}=130$

40 $4(m-9)(n-1)+4(m-8)(n-2)+4(m-7)(n-3)+4(m-6)(n-4)$
$=\overset{9,1}{16}+\overset{8,2}{24}+\overset{7,3}{24}+\overset{6,4}{16}=\ 80$

41 $4(m-9)(n-2)+4(m-8)(n-3)+4(m-7)(n-4)=\overset{9,2}{12}+\overset{8,3}{16}+\overset{7,4}{12} \qquad =\ 40$

42 $4(m-9)(n-3)+4(m-8)(n-4)=\overset{9,3}{8}+\overset{9,3}{8} \qquad =\ 16$

43 $4(m-9)(n-4)=\overset{9,4}{4} \qquad =\ 4$

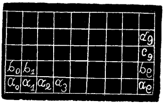
The law of formation is apparent.

But each row may zigzag in many ways and occupy many positions. Let the figure, for the present, represent any layer of balls and let us consider the ways of passing from a_0 to b_9, c_9, etc.

To pass to b_9 we may jog upward at a_0 a_1 .... or a_9 giving 10 ways.

To pass to c_9 if we jog upward one space at a_0 (to b_0), then as just shown there are 10 ways to c_9; if the jog is at a_1 (to b_1) there are 9 ways to c_9; if the jog is at a_2 (to b_2) there are 8 ways to c_9, etc. Hence from a_0 to c_9 there are $10\times9+\ldots.+1=\frac{11\times10}{1.2}=C_2^{11}$ ways. To pass to d_9 if we jog from a_0 to b_0 then from b_0 to d_9 is the same as from a_0 to c_9 or as just shown, C_2^{11} ways.

From b_1 to d_9 there are for same reason $9+8+\ldots.+1=\frac{10+9}{1.2}=C_2^{10}$ ways. etc., etc.



Hence in all there are

$C_2^{11}+C_2^{10}+\ldots.+C_2^2=C_3^{12}=C_9^{12}$ ways.

Hence, in general, if m=the sum of the number of spaces or steps in both directions between the centers of the first and the last ball, and n= the number in either direction, the number of ways from the first to the last $=C_n^m$. Observe that the number of balls $=m+1$.

Now consider rows that begin and end in different layers.

Let N=the number of balls in any row. Consider two layers. Suppose *any* row beginning and ending at given points in adjacent horizontal layers to be projected on the lower layer.

There are $N-1$ balls and $N-2$ spaces in the projected row; and hence $N-2-29=N-31$ spaces latterally. Hence the number of projected rows is C_{N-31}^{N-2}. It is necessary to find the number of ways each projected row (and therefore the original row) can be divided between the two layers. To do this we observe that the real length of the row is the length of the box added to the latteral deviation of the row.

Since there are $N-2$ spaces horizontally and one space vertically, the number of ways of distributing the N balls of any projected row on the two layers is, $C_1^{N-2+1}=C_1^{N-1}$.

Hence any two given points on opposite ends of two adjacent layers can be connected with N ball in $C_1^{N-1}.C_{N-31}^{N-2}$ ways.

$$\text{Now } C_1^{N-1}.C_{N-31}^{N-2}=\frac{N-1}{1}.\frac{(N-2)(N-3)\ldots.30}{1.\ 2.\ 3\ldots.(N-31)}.$$

$$\text{But } C_{N-30}^{N-1}=\frac{(N-1)(N-2)\ldots.30}{1.2.3\ldots.(N-31)(N-30)}$$

$$\text{Hence } C_1^{N-1}.C_{N-31}^{N-2}=N-30\,C_{N-30}^{N-1}=C_1^{N-30}.C_{N-30}^{N-1} \text{ ways.}$$

Let there be three layers.

Suppose any row to be projected upon the lower layer. There are $N-2$ balls and $N-3$ spaces horizontally. Also two spaces vertically and $N-3-29=N-32$ latterally.

Hence the total number of projected rows is, C^{N-3}_{N-32}. Each row can be distributed in, $C_2^{N-3+2}=C_2^{N-1}$ ways. Hence any two given points on opposite ends of the first and last of three adjacent layers can be connected with N balls in $C_2^{N-1}.\,C^{N-3}_{N-32}$ ways.

$$\text{Now } C_2^{N-1}.\,C^{N-3}_{N-32}=\frac{(N-1)(N-2)}{1.2}\times\frac{(N-3)(N-4)\dots.30}{1.2.3\dots..N-32}$$

$$\text{and } C^{N-1}_{N-30}=\frac{(N-1)(N-2)(N-3)(N-4)\dots.30}{1.2.3\dots..(N-32)(N-31)(N-30)}.$$

$$\therefore\quad C_2^{N-1}.\,C^{N-3}_{N-32}=\frac{(N-30)(N-31)}{1.2}.\,C^{N-1}_{N-30}=C_2^{N-30}.\,C^{N-1}_{N-30}\text{ ways.}$$

Similarly for four adjacent layers there are, $C_3^{N-30}.\,C^{N-1}_{N-30}$ ways,

and for five adjacent layers, $C_4^{N-30}.\,C^{N-1}_{N-30}$ ways.

For n adjacent layers there are, $C^{N-30}_{N-1}.\,C^{N-1}_{N-30}$ ways.

Now out of 750 white balls 30 may be drawn in,

$$\frac{750.749\dots.721}{1.2.3\dots.30}=a\text{ ways.}$$

Also $1500-30=1470$ may be drawn in $\underline{|1470}=b$ ways.

$$31\text{ white balls may be drawn in }\frac{750.748\dots.720}{1.2.3\dots.31}=a_1\text{ ways, and}$$

$1500-31$ may be drawn in $\underline{|1469}=b_1$ ways. Similarly for other numbers.

Represent the numbers of ways of drawing 32,33....43 balls by a_1, $a_3\dots.a_{13}$ and the ways of drawing the complementary numbers 1468,1467 ....1457 by b_2, $b_3\dots.b_{13}$.

Now we have as follows:

30 white balls may be drawn in a ways, 1470 in b ways. There are of 30 balls each 50 rows, and each row has $C_0^{29}=1$ position.

Hence such a row may be formed in 50 $C_0^{29}\,ab=50\,ab$ ways.

Since there are 170 rows of 31 ball each and each row may take C_1^{30} positions such a row may be formed in 170 $C_1^{30}\,a_1\,b_1=5100\,a_1\,b_1$ ways.

There are 120 rows of 32 balls each in the same layer, and 144 rows occupying adjacent layers, each one of the first may take C_2^{31} positions and each one of the second set 2 C_2^{31} positions.

Hence such a row may be formed in $(140+2.144)\,C_2^{31}a_2b_2=428\,C_2^{31}a_2b_2$ ways.

There are 110 rows of 33 balls each in the same layer and 236 occupying adjacent layers.

Hence such a row may be formed in $(110+3.236)\,C_3^{32}a_3b_3$ $=818\,C_3^{32}a_3b_3$ ways. There are 80 rows of 34 balls each in the same layer, 184 occupying two layers and 96 occupying three layers.

Hence such a row may be formed in, $(80+4.184+6.96)C_4^{33}a_4b_4 = 1392C_4^{33}a_4b_4$ ways.

In a similar manner we find,

For 35 balls, $(50+5.132+10.148)C_5^{34}a_5b_5=219C_5^{34}a_5b_5$ ways.

For 36 balls, $(40\times6.80+15.104+20.56)C_6^{35}a_6b_6=3200C_6^{35}a_6b_6$ ways.

For 37 balls, $(0+7.64+21.60+35.76)C_7^{36}a_7b_7=4438C_7^{56}a_7b_7$ ways.

For 38 balls, $(20+8.48+28.48+56.40+70.24)C_8^{37}a_8b_8=5668C_8^{37}a_8b_8$ ways.

For 39 balls, $(10+9.32+36.36+84.32+126.20)C_9^{38}a_9b_9=6802C_9^{38}a_9b_9$ ways.

For 40 balls, $(10.16+45.24+120.24+210.16)C_{10}^{39}a_{10}b_{10}=7480C_{10}^{39}a_{10}b_{10}$ ways.

For 41 balls, $(55.12+165.16+330.12)C_{11}^{40}a_{11}b_{11}=7260C_{11}^{40}a_{11}b_{11}$ ways.

For 42 balls, $(220.8+495.8)C_{12}^{41}a_{12}b_{12}=5720C_{12}^{41}a_{12}b_{12}$ ways.

For 43 balls, $715.4C_{13}^{42}a_{13}b_{13}=2860C_{13}^{42}a_{13}b_{13}$ ways.

Let n, n_1, $n_2 \dots n_{13}$ represent the numbers of ways of forming the different rows as above shown and let N_1 represent the total number of ways of drawing the balls. Thus $N_1=\underline{|1500}$, and the probability sought is,

$$p=\frac{n+n_1+n_2+\dots n_{13}}{N_1}.$$

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

ANSWER TO A QUERY IN JUNE MONTHLY.

On page 214 of the *American Mathematical Monthly* Vol. I. No. 6., appears a Query in regard to the work of Giordano da Bitonto: Euclide restituto overe gli antichi elementi geometrici ristoranti; Roma, 1680. Folio.

In answer to this Dr. Halsted has received a letter from the celebrated Gino Loria of the University of Genoa, of which the following is a translation:

GENOA, AUGUST 27th 1894.

HONORED COLLEAGUE,

Having read the question which you have proposed at p. 214 of the *American Mathematical Monthly*, I have asked Mr. Ricardi, former professor of the University of Modena, to give a reply to it. He has answered that the work of Giordano di Bitonto which interests you, and even a falsification which appeared in 1681 exist in his own rich private library. Both are described in the great work of this professor intitled *Bibliotea matematici italiana*, of which a second edition is now in press I hope that these data will suffice you. If not, address me, and I will be glad to furnish the information you desire.

I await the continuation of your interesting article on the non-Euclidean geometry.

Accept, dear sir, the expression of my distinguished consideration.

Your very devoted,

GINO LORIA.

RESOLUTIONS BY THE FRENCH MATHEMATICIANS AT THE CAEN CONGRESS.

TRANSLATED BY GEORGE BRUCE HALSTED.

French Association for the advancement of Science; Congress of Caen, 1894; Sections First and Second. Seance of August 14th, 1894.

The question for the order of the day was, a study of means to facilitate a more facile and fruitful exchange of ideas between the mathematicians of diverse nations, and thus to contribute to the progress of the science and the perfecting of its methods.

After a thorough discussion in which a large number of members took part, the following RESOLUTIONS were passed unanimously:

1 We approve most completely the project to establish *international mathematical Congresses,* and declare ourselves desirous of aiding the efforts which are making or shall be made toward this aim;

2 We approve absolutely the idea of Mr. Mansion, relative to the making of *Mathematical Vocabularies*, and applaud the beginning of its realisation that Commandant Brocard has already given by the preparation of a French Mathematical Vocabulary:

3 We express the hope that the project of Mr. Jacques Boyer, for a *Mathematical Dictionary* will reach a favorable issue, both in France and in most other countries;

4 We think we should direct attention to the remarkable mathematical monographs now being published in Germany, which it would be very desirable to see translated into other languages;

5 We think the great efforts of Professor Peano and his confreres for the propagation of *Algorithmic Logic* and the publication of a mathematical formulary are of a nature to contribute powerfully to the aim whose attainment is now in question;

6 We are happy to mention the decided advancement of the *Repertoire bibliographique des Sciences mathematiques*, and in this connection, to applaud the interesting publication, due to a group of mathematicians in Holland, especially Mr. P. H. Schoute, which is entitled *Revue Semestrielle des Publications Mathematiques*;

7 We think that the publication of the *Intermediare des Mathematiciens*, from the beginning of 1894, has rendered and will still render very great service in what concerns the relations of Mathematicians to one another; we express our gratitude to the founders, Laisant and Lemoine, and felicitate ourselves that this initiative is due to two members of the French Association for the advancement of Science;

8 We hold in very serious consideration the reflections presented by Mr. Lemeray on the possibility of establishing mathematical libraries, having for object to put books at the disposal of workers distant from scientific centers;

9 We decide that this question for the order of the day, in the general form into which it has been put, shall be maintained for the session at Bordeaux in 1895.

EDITORIALS.

At an informal meeting of the Texas Academy of Science, October 12, the president, George Bruce Halsted, gave his anaugural address, the subject being: *Original research and creative authorship the essence of University teaching.*

On page 250 of the MONTHLY, the notice of *The Mathematical Magazine* should read: Issued at irregular intervals. Price, $1.00 in advance for four numbers.

Principal A. J. Lilly, of the Northern Iowa Normal and Commercial School, Angola, Iowa, says, All the numbers of the MONTHLY are good and contain much valuable information.

Dr. George Lilley of Portland, Oregon, has a complete set of the *Analyst* which he desires to sell. Anyone desiring a complete set of this valuable Journal will do well to write to Dr. Lilley. Mr. L. B. Fraker of Weston, Ohio, has all but three Nos. of the *Analyst* which he would be pleased to sell.

E. D. West, of West Middleburg, Ohio, expresses himself as being well pleased with the MONTHLY.

We have on hand a number of criticisms to leading articles in the MONTHLY, also a number of replies to previous controversies; but as we also have numerous papers of high order and great value, we desire that these shall appear first.

BOOKS AND PERIODICALS.

A New Life in Education. By Fletcher Durell, Ph. D., professor in Dickinson College, is published by the American Sunday-School Union, Philadelphia, Pa., as No. 9a of the Green Fund Books.

"This work received the first prize of six hundred dollars offered by The American Sunday School Union, under the provision of the John C. Green Income Fund."

The book is a 12 mo. with 288 pages. It is divided into twelve chapters besides an appendix.

"The book is written", as the author says in his preface, with a double purpose, to discuss, first, the place of the religious element in education; and second, the place and function of the highest type of education, in the immediate future."

It is written in an easy, clear, and fluent style, and so facinating that it is difficult to lay it down until you have completed it.

It is one of those pure wholesome books that deserve to have a wide circulation.

It should be placed in the hands of every teacher and pupil.

G. W. SHAW.

El Progresso Mathematico. Periodico de Mathematicas Puras y Aplecadas. Director Don Loel G. de Galdeano, Catedratico de la Universidad de Zaragoza.

The editor of El Progresso Mathematico has just sent us a copy of each of the eight issues for the year 1894. This journal is published monthly and is devoted to the solutions of problems in pure and applied mathematics. Many papers on interesting mathematical subjects also appear. The journal is well printed and many of the solutions are illustrated with beautiful black diagrams. The price of the journal to subscribers within the limits of the Postal Union is 11 fr. B. F. F.

Nicolai Ivanovich Lobaachevsky. Address. Pronounced at the Commemorative meeting of the Imperial University of Kasan, October 22, 1893. By Professor A. Vasiliev, President of the Physico-Mathematical Society of Kasan. Translate from the Russian, with a preface by Dr. George Bruce Halsted, President of the Texas Academy of Science.

We read this very excellent address with much pleasure and profit. Many parts of the address are very eloquent and througout is interesting and instructive. Those who are unfamiliar with the Russian language owe Dr. Halsted a debt of gratitude for making known to them the best Russian thought of the 19th century.

Modern Analytical Geometry.

An Introductory Account of Certain Modern Ideas and Methods in Plane Analytical Geometry. By Charlotte Angas Scott, D. Sc., Girton College, Cambridge; Professor of Mathematics in Bryn Mawr College, Pennsylvania. Large 8vo Cloth. XII+288pp. ($5\frac{1}{2}$x9). Price, $2.50. New York: Macmillan & Co.

A glance at the subjects treated will give an idea of the nature of this work. Chapter I. treats of Point and Line Coordinates; Chapter II., Infinity, Transformation of Coordinates; Chapter III., Figures Determined by Four Elements; Chapter IV., the principal of Duality; Chapter V., Descriptive Properties of Curves; Chapter VI., Metrical Properties of Curves; the Line-Infinitiy; Chapter VIII., Unicursal Curves. Tracing of Curves; Chapter IX., Cross-Ratio, Homography, and Involution; Chapter X., Projection and Linear Transformation; Chapter XII., The Absolute; Chapter XIII., Invariants and Covariants.

The book is well written and is a fine specimen of eminent scholarship. It will receive a hearty welcome from all mathematicians who are desirous to acquaint themselves with the progress made in Modern Higher Mathematics. B.F.F.